Exact results for deterministic cellular automata traffic models

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We present a rigorous derivation of the flow at arbitrary time in a deterministic cellular automaton model of traffic flow. The derivation employs regularities in preimages of blocks of zeros, reducing the problem of preimage enumeration to a well-known lattice path counting problem. Assuming infinite lattice size and random initial configuration, the flow can be expressed in terms of generalized hypergeometric function. We show that the steady-state limit agrees with previously published results. [S1063-651X(99)07207-4]

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I. INTRODUCTION

Since the introduction of the Nagel-Schreckenberg (NS) model in 1992 [1], cellular automata became a well-established method of traffic flow modeling. Comparatively low computational cost of cellular automata models made it possible to conduct large-scale real-time simulations of urban traffic in the city of Duisburg [2] and Dallas/Forth Worth [3]. Several simplified models have been proposed, including models based on deterministic cellular automata. For example, Nagel and Herrmann [4] considered deterministic version of the NS model, while Fukui and Ishibashi (FI) [5] introduced another model, which can be understood as a generalization of cellular automaton rule 184. Rule 184, one of the elementary Cellular automaton (CA) rules investigated by Wolfram [6], had been later studied in detail as a simple model of surface growth [7], as well as in the context of density classification problem [8]. It is one of the only two (symmetric) nontrivial elementary rules conserving the number of active sites [9], and, therefore, can be interpreted as a rule governing dynamics of particles (cars). Particles (cars) move to the left if their right neighbor site is empty, and do not move if the right neighbor site is occupied, all of them moving simultaneously at each discrete time step. Using terminology of lattice stochastic processes, rule 184 can be viewed as a discrete-time version of totally asymmetric simple exclusion process. Further generalization of the FI model has been proposed in [10].

In all traffic models, the main quantity of interest is the average velocity of cars, or the average flow, defined as a product of the average velocity and the density of cars. The graph of the flow as a function of density is called a fundamental diagram, and is typically studied in the steady state ($t\to\infty$). For the FI model, a steady-state fundamental diagram can be obtained using mean-field argument [5], as well as by statistical mechanical approach [11] or by studying the time evolution of intercar spacing [12]. In general, little is known about nonequilibrium properties of the flow. In [8], we investigated dynamics of rule 184 and derived expression for the flow at arbitrary time, assuming that the initial configuration (at $t=0$) was random, using the concept of defects and analyzing the dynamics of their collisions. In what follows, we shall generalize results of [8] for the deterministic FI traffic flow model and derive the expression for the flow at arbitrary time. The derivation employs regularities of preimages of blocks of zeros, reducing the problem of preimage enumeration to a well-known combinatorial problem of lattice path counting. Assuming infinite lattice size and random initial configuration, the flow can then be expressed in terms of generalized hypergeometric function. We will, unlike in [8], explore regularities of preimages using purely algebraic methods, i.e., without resorting to properties of spatiotemporal diagrams and dynamics of defects.

II. DETERMINISTIC TRAFFIC RULES

Deterministic version of the FI traffic model is defined on a one-dimensional lattice of L sites with periodic boundary conditions. Each site is either occupied by a vehicle, or empty. The velocity of each vehicle is an integer between 0 and m. If $x(i,t)$ denotes the position of the ith car at time t, the position of the next car ahead at time t is $x(i+1,t)$. With this notation, the system evolves according to a synchronous rule given by

$$x(i,t+1)=x(i,t)+v(i,t), \tag{1}$$

where

$$v(i,t)=\min(x(i+1,t)-x(i,t)-1,m) \tag{2}$$

is the velocity of car i at time t. Since $g=x(i+1,t)-x(i,t)-1$ is the gap (number of empty sites) between cars i and $i+1$ at time t, one could say that each time step, each car advances by g sites to the right if $g\leqslant m$, and by m sites if $g>m$. When $m=1$, this model is equivalent to elementary cellular automaton rule 184, for which a number of exact results is known [7,8].

The main quantities of interest in this paper will be the average velocity of cars at time t defined as

$$\overline{v(t)}=\frac{1}{N}\sum_{i=1}^{N} v(i,t), \tag{3}$$

and the average flow $\phi(t)=\rho\overline{v(t)}$, where $\rho=N/L$ is the density of cars. In what follows, we will assume that at $t=0$ the

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cars are randomly distributed on the lattice. When $N\rightarrow\infty$, this corresponds to a situation when sites are occupied by a car with probability ρ, or are empty with probability $1-\rho$.

In general, if $N_k(t)$ is the number of cars with velocity k, we have

$$\overline{v}(t)=\frac{1}{N}\sum_{k=1}^{m} kN_k(t). \tag{4}$$

When $k<m$, $N_k(t)$ is just the number of blocks of type 10^k1, where 0^k denotes k zeros. This means that a probability of an occurrence of the block 10^k1 at time t can be written as $P_t(10^k1)=N_k/L$. Similarly, for $k=m$, $P_t(10^m)=N_m(t)/L$. As a consequence, Eq. (4) becomes

$$\overline{v}(t)=\sum_{k=1}^{m-1}\frac{kP_t(10^k1)}{\rho}+\frac{mP_t(10^m)}{\rho}. \tag{5}$$

We will now demonstrate that in the deterministic FI model with maximum speed m the average flow depends only on one block probability. More precisely, we shall prove the following:

Proposition 1. In the deterministic FI model with the maximum speed m, the average flow $\phi_m(t)$ is given by

$$\phi_m(t)=1-\rho-P_t(0^{m+1}). \tag{6}$$

To prove this proposition by induction, we first note that for $m=1$ Eq. (5) gives $\phi_1(t)=P_t(10)$. Using consistency condition for block probabilities $P_t(10)+P_t(00)=P_t(0)=1-\rho$, we obtain $\phi_1(t)=1-\rho-P(00)$, which verifies Eq. (6) in the $m=1$ case. Now assume that Eq. (6) is true for some $m=n-1$ (where $n>1$), and compute $\phi_n(t)$:

$$\begin{aligned}\phi_n(t)&=nP_t(10^n)+\sum_{j=1}^{n-1} jP(10^j1)\\ &=nP_t(10^n)+(n-1)P_t(10^{n-1}1)+\sum_{j=1}^{n-2} jP(10^j1)\\ &=(n-1)[P_t(10^{n-1}1)+P_t(10^n)]+P_t(10^n)\\ &\quad+\sum_{j=1}^{n-2} jP(10^j1).\end{aligned}$$

Using the consistency condition $P_t(10^{n-1}1)+P_t(10^n)=P_t(10^{n-1})$ we obtain

$$\begin{aligned}\phi_n(t)&=P_t(10^n)+(n-1)P_t(10^{n-1})+\sum_{j=1}^{n-2} jP(10^j1)\\ &=P_t(10^n)+\phi_{m-1}(t).\end{aligned}$$

Taking into account that $P_t(10^n)=P_t(0^n)-P_t(0^{n+1})$ (which, again, is just a consistency condition for block probabilities), and using Eq. (6) to express $\phi_{m-1}(t)$, we finally obtain

$$\phi_m(t)=1-\rho-P_t(0^{m+1}). \tag{7}$$

This means that the validity of Eq. (6) for $m=n$ follows from its validity for $m=n-1$, concluding our proof by induction.

III. ENUMERATION OF PREIMAGES OF 0^{m+1}

Proposition 1 reduces the problem of computing $\phi_m(t)$ to the problem of finding the probability of a block of $m+1$ zeros. In order to find this probability, we will now use the fact that the deterministic FI model is equivalent to a cellular automaton defined as follows. Let $s(i,t)$ denote the state of a lattice site i at time t (note that i now labels consecutive lattice sites, *not* consecutive cars), where $s(i,t)=1$ for a site occupied by a car and $s(i,t)=0$ otherwise. We can immediately realize that if a site i is empty at time t, then at time $t+1$ it can become occupied by a car arriving from the left, but not from a site further than $i-m$. Similarly, if a site i is occupied, it will become empty at the next time step only and only if site $i+1$ is empty. Thus, in general, $s(i,t+1)$ depends on $s(i-m,t),s(i-m+1,t),\ldots,s(i+1,t)$, i.e., on the state of m sites to the left, one site to the right, and itself, but not on any other site, that can be expressed as

$$s(i,t+1)=f_m(s(i-m,t),s(i-m+1,t),\ldots,s(i+1,t)), \tag{8}$$

where f_m is called a local function of the cellular automaton. For the FI CA, one can write explicit formula[1] for f_m, such as

$$\begin{aligned}&f_m(s(i-m,t),s(i-m+1,t),\ldots,s(i+1,t))\\ &\quad=s(i,t)-\min\{s(i,t),1-s(i+1,t)\}\\ &\quad\quad+\min\{\max[s(i-m,t),s(i-m+1,t),\ldots,\\ &\quad\quad s(i-1,t)],1-s(i,t)\},\end{aligned} \tag{9}$$

which, using terminology of cellular automata theory, represents a rule with left radius m and right radius 1. In general, after t iteration of this cellular automaton rule, state of a site $s(i,t)$ depends on $s(i-mt,0),s(i-mt+1,0),\ldots,s(i+t,0)$, but not on any other sites in the initial configuration. Similarly, a block of k sites $s(i,t)s(i+1,t)\cdots s(i+k)$ depends only on a block $s(i-mt,0),s(i-mt+1,0),\ldots,s(i+k+t,0)$, as schematically shown in Fig. 1. We will say that $s(i-mt,0),s(i-mt+1,0),\ldots,s(i+k+t,0)$ is a t-step preimage of the block $s(i,t)s(i+1,t)\cdots s(i+k)$. Preimages in the FI cellular automaton have the following property:

Proposition 2. Block $a_1a_2a_3\cdots a_p$ is an n-step preimage of a block 0^{m+1} if and only if $p=(n+1)(m+1)$ and, for every k $(1\leqslant k\leqslant p)$,

$$\sum_{i=1}^{k}\xi(a_i)>0, \tag{10}$$

where $\xi(1)=-m$ and $\xi(0)=1$.

[1] Since formula (9) will not be used in subsequent calculations, we give it without proof (which is elementary).

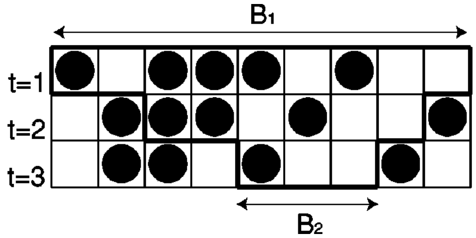

FIG. 1. Fragment of a spatiotemporal diagram for the FI rule with $m=2$. States of nine sites during three consecutive time steps are shown, black circles representing occupied sites. Block $B_1=101110100$ is a two-step preimage of the block $B_2=100$. Outlined sites constitute ''light cone'' of the block B_2, meaning that the state of sites belonging to B_2 can depend only on sites inside the outlined region, but not on sites outside this region.

Before we present a proof of this proposition, note that it can be interpreted as follows. Let us assume that we have a block of zeros and ones of length p, where $p=(n+1)(m+1)$, and we want to check if this block is an n-step preimage of a block 0^{m+1}. We start with a ''capital'' equal to zero. Now we move from the leftmost site to the right, and every time we encounter 0, we increase our capital by m. Every time we encounter 1, our capital decreases by 1. If we can move from a_1 to a_p and our capital stays always larger than zero, the string $a_1a_2a_3\ldots a_p$ is a preimage of 0^{m+1}. Condition (10) can be also written as

$$\sum_{i=1}^{k} a_i < \frac{k}{m+1}, \tag{11}$$

because $\xi(x)=1-(m+1)x$ for $x\in\{0,1\}$.

For the purpose of the proof, strings $a_1a_2\cdots a_p$ of length p satisfying Eg. (11) for a given m and for every $k\leqslant p$ will be called m-admissible strings.

Lemma. Let $s(1,t)s(2,t)\cdots s(p,t)$ be an m-admissible string. If

$$s(i,t+1)=f_m(s(i-m,t),s(i-m+1,t),\ldots,s(i+1,t)), \tag{12}$$

and if f_m is a local function of the deterministic FI model with maximum speed m, then $s(m+1,t+1)s(m+2,t+1)\cdots s(p-1,t+1)$ is also an m-admissible string.

To prove the lemma, it is helpful to employ the fact that the FI rule conserves the number of cars. Let $0<k<p$ and let us consider strings $S_1=s(1,t)s(2,t)\cdots s(k,t)$ and $S_2=s(m+1,t+1)s(2,t+1)\cdots s(k,t+1)$. If the string $s(1,t)s(2,t)\cdots s(k,t)$ is m-admissible, then its first $m+1$ sites must be zeros. This means that in one time step, no car can enter string $s(1,t)s(2,t)\cdots s(k,t)$ from the left. On the other hand, in a single time step, only one car (or none) can leave the string on the right-hand side, i.e.,

$$\sum_{i=1}^{k} s(i,t)=\epsilon+\sum_{i=m+1}^{k} s(i,t+1), \tag{13}$$

where $\epsilon\in\{0,1\}$. Three cases can be distinguished:

(i) All sites $s(k-m+1,t)s(k-m+2,t)\cdots s(k,t)$ are empty (equal to 0). Then no car leaves S_1, which means that $\epsilon=0$, and

$$\sum_{i=1}^{k} s(i,t)=\sum_{i=m+1}^{k} s(i,t+1)=\sum_{i=m+1}^{k-m} s(i,t+1)<\frac{k-m}{m+1}. \tag{14}$$

The last inequality is a direct consequence of m admissibility of S_1. Since the length of the string S_2 is equal to $k-m$, the above relation (which holds for arbitrary k) proves that S_2 is also m admissible in the case considered.

(ii) Among sites $s(k-m+1,t)s(k-m+2,t)\cdots s(k,t)$ there is at least one which is occupied (equal to 1), and $s(k+1,t)=1$. In this case, since the last site in S_1 is ''blocked'' by the car at $s(k+1,t)$, again no car can leave string S_1 in one time step. Therefore,

$$\sum_{i=1}^{k} s(i,t)=\sum_{i=m+1}^{k} s(i,t+1). \tag{15}$$

m-admissibility of S_2 implies

$$\frac{k+1}{m+1}>\sum_{i=1}^{k+1} s(i,t)=\sum_{i=1}^{k} s(i,t)+1. \tag{16}$$

Combining Eq. (15) with Eq. (16) we obtain

$$\sum_{i=m+1}^{k-m} s(i,t+1)<\frac{k-m}{m+1}, \tag{17}$$

which again shows that S_2 is m admissible.

(iii) Among sites $s(k-m+1,t)s(k-m+2,t)\cdots s(k,t)$ there is at least one which is occupied (equal to 1), and $s(k+1,t)=0$. In this case, one car will leave at the right end of the string S_1; therefore,

$$\sum_{i=1}^{k} s(i,t)=\sum_{i=m+1}^{k} s(i,t+1)-1. \tag{18}$$

As before, from m admissibility of S_1 we have

$$\sum_{i=1}^{k+1} s(i,t)=\sum_{i=1}^{k} s(i,t)<\frac{k+1}{m+1}; \tag{19}$$

hence,

$$\sum_{i=m+1}^{k} s(i,t+1)=\sum_{i=m+1}^{k} s(i,t+1)-1<\frac{k+1}{m+1}-1=\frac{k-m}{m+1}, \tag{20}$$

which demonstrates that case (iii) also leads to m admissibility of S_2, concluding the proof of our lemma.

Let us now assume that the block $B_1=s(1,t)s(2,t)\cdots s(p,t)$ is m admissible [n being some fixed integer and $p=(m+1)(n+1)$]. Applying the lemma to this block we conclude that $B_2=s(m+1,t+1)s(2,t+1)\cdots s(p-1,t+1)$ is m admissible as well. Applying the lemma to B_2 we obtain m-admissible block $B_3=s(2m+1,t+2)s(2,t+2)\cdots s(p-2,t+2)$. After n applications of the lemma we end up with the conclusion that the string $B_{n+1}=s(nm+1,n+1)s(nm+2,n+1)\cdots s(p-n)$ is m admissible. Since the length of B_{n+1} is $p-n-nm=(n+1)(m+1)-n(m+1)=m+1$, it must, to be m admissible, be composed of all

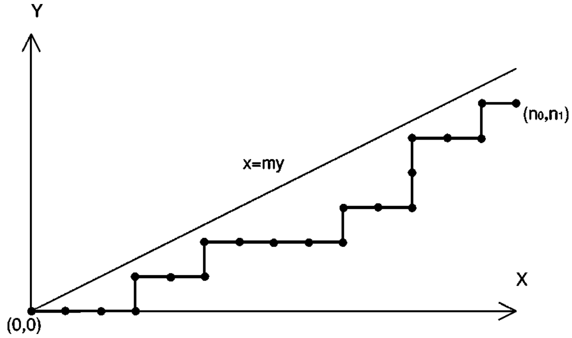

FIG. 2. m-admissible block with n_0 zeros and n_1 ones is equivalent to a lattice path from the origin to (n_0,n_1), which does not touch nor cross the line $x=my$. 0 corresponds to a horizontal segment, while 1 corresponds to a vertical segment.

zeros, i.e., $B_{m+1}=0^{m+1}$. This means that m admissibility of B_1 is a sufficient condition for B_1 to be an n-step preimage of 0^{m+1}. Reversing steps in the above reasoning, one can show that it is also a necessary condition.

IV. FUNDAMENTAL DIAGRAM

We shall now use Proposition 2 to calculate $P_t(0^{m+1})$. First of all, we note that $P_t(0^{m+1})$ is equal to the probability of occurrence of t-step preimage of 0^{m+1} in the initial (random) configuration; that is,

$$P_t(0^{m+1})=\sum P_0(a), \tag{21}$$

where the sum goes over all t-step preimages of 0^{m+1}. Consider now a string, which contains n_0 zeros and n_1 ones. The number of such strings can be immediately obtained if we realize that it is equal to the number of lattice paths from the origin to (n_0,n_1) that do not touch nor cross the line $x=my$, as shown in Fig. 2. This is a well-known combinatorial problem [13], and the number of aforementioned paths equals

$$\frac{n_0-mn_1}{n_0+n_1}\binom{n_0+n_1}{n_1}. \tag{22}$$

Probability of occurrence of such a block in a random configurations is, therefore,

$$\frac{n_0-mn_1}{n_0+n_1}\binom{n_0+n_1}{n_1}\rho^{n_1}(1-\rho)^{n_0}, \tag{23}$$

where $\rho=P(1)$. In a t-step preimage of 0^{m+1} the minimum number of zeros is $1+m(t+1)$, while the maximum is $(m+1)(t+1)$ (corresponding to all zeros). Therefore, summing over all possible numbers of zeros i, we obtain

$$P_t(0^{m+1})=\sum_{i=1+m(t+1)}^{(m+1)(t+1)}\frac{i-m[(m+1)(t+1)-i]}{(m+1)(t+1)} \times\binom{(m+1)(t+1)}{(m+1)(t+1)-i}\times\rho^{(m+1)(t+1)-i}(1-\rho)^i.$$

Changing summation index $j=i-m(t+1)$ we obtain

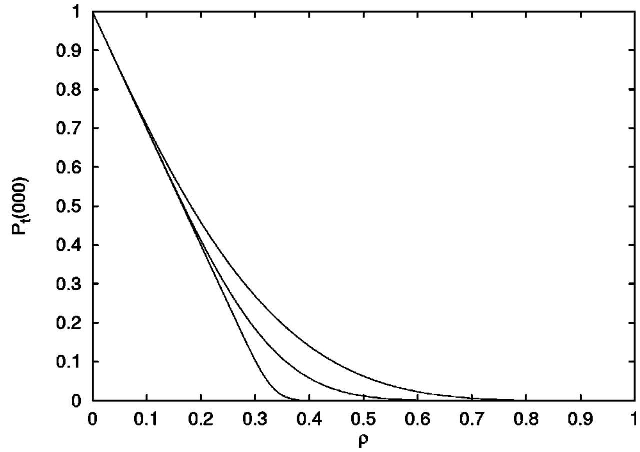

FIG. 3. Graph of the probability $P_t(0^{m+1})$ as a function of ρ for $m=2$ and $t=1$ (upper line), $t=5$ (middle line), and $t=100$ (lower line).

$$P_t(0^{m+1})=\sum_{j=1}^{t+1}\frac{j}{t+1}\binom{(m+1)(t+1)}{t+1-j} \times\rho^{t+1-j}(1-\rho)^{m(t+1)+j}. \tag{24}$$

Figure 3 shows a graph of $P_t(0^{m+1})$ as a function of ρ for $m=2$ and several values of t. We can observe that as t increases, the graph becomes ''sharper'' at $\rho=1/3$, eventually developing singularity (discontinuity in the first derivative) at $\rho=1/3$. More precisely, one can show (see Appendix) that

$$\lim_{t\to\infty}P_t(0^{m+1})=\begin{cases}1-(m+1)\rho & \text{if } p<1/(m+1)\\ 0 & \text{otherwise.}\end{cases} \tag{25}$$

$P_\infty(0^{m+1})$, therefore, can be viewed as the order parameter in a phase transition with critical point at $\rho=1/(m+1)$. Using Proposition 1 we can now find the average flow in the steady state

$$\phi_m(\infty)=\begin{cases}m\rho & \text{if } p<1/(m+1)\\ 1-\rho & \text{otherwise,}\end{cases} \tag{26}$$

which agrees with mean-field type calculations reported in [5] as well as with results of [11,12]. To verify validity of the result for $t<\infty$, we performed computer simulations using a lattice of 10^5 sites with periodic boundary conditions. The average flow has been recorded after each iteration up to $t=100$ for three values of ρ: at the critical point $\rho=1/3$ as well as below and above the critical point. The resulting plots of the flow as a function of time are presented in Fig. 4. Again, the agreement with theoretical curves,

$$\phi_m(t)=1-\rho-\sum_{j=1}^{t+1}\frac{j}{t+1}\binom{(m+1)(t+1)}{t+1-j} \times\rho^{t+1-j}(1-\rho)^{m(t+1)+j}, \tag{27}$$

is very good. Without going into details, we note that the formula (27) can be also expressed in terms of generalized hypergeometric function ${}_2F_1$:

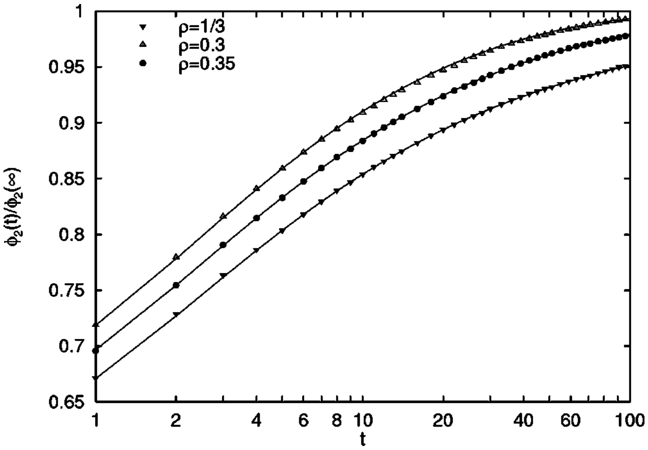

FIG. 4. Plots of $\phi_2(t)/\phi_2(\infty)$ as a function of time for $\rho=0.3$, $\rho=1/3$, and $\rho=0.35$ obtained from computer simulation on a lattice of 10^5 sites. Continuous line corresponds to the theoretical result obtained using Eq. (28).

$$\phi_m(t)=1-\rho-\frac{(1-\rho)^{1+m+mt}\rho^t(1+m+t+mt)!}{(1+m+mt)(1+t)!(m+mt)!}\times{}_2F_1\left[\begin{matrix}2,-t\\2+m+mt\end{matrix};1-\frac{1}{\rho}\right]. \tag{28}$$

Since fast numerical algorithms for computing ${}_2F_1$ exist, this form might be useful for the purpose of numerical evaluation of $\phi_m(t)$.

V. CONCLUSION

We presented derivation of the flow at arbitrary time in the deterministic FI cellular automaton model of traffic flow. First, we showed that the flow can be expressed by the probability of occurrence of the block of $m+1$ zeros $P(0^{m+1})$. By employing regularities in preimages of blocks of zeros, we reduced the problem of preimage enumeration to the lattice path counting problem. Finally, we used the number of preimages to find $P(0^{m+1})$, which determines the flow.

We also found that the flow in the steady state, obtained by taking $t\to\infty$ limit, agrees with previously reported mean-field-type calculations, meaning that in the case of the FI model mean-field approximation gives exact results. This seems to be true not only for the FI model, but also for many other CA rules conserving the number of active sites (‘‘conservative’’ CA). For example, in [9] we reported that the third order local structure approximation, which is a generalization of simple mean-field theory incorporating short-range correlations, yields the fundamental diagram for rule 60 200 (one of the 4-input ‘‘conservative’’ CA rules) in extremely good agreement with computer simulations. Taking this into account, we conjecture that the local structure approximation gives an exact fundamental diagram for almost all ‘‘conservative’’ rules, excluding, perhaps, those rules for which the fundamental diagram is not sufficiently ‘‘regular’’ (meaning not piecewise linear). This problem is currently under investigation.

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APPENDIX

In order to find the limit $\lim_{t\to\infty} P_t(0^{m+1})$ we can write Eq. (24) in the form

$$P_t(0^{m+1})=\sum_{j=1}^{t+1}\frac{j}{t+1}b(t+1-j,(m+1)(t+1),\rho), \tag{A1}$$

where

$$b(k,n,p)=\binom{n}{k}p^k(1-p)^{n-k} \tag{A2}$$

is the distribution function of the binomial distribution. Using de Moivre-Laplace limit theorem, binomial distribution for large n can be approximated by the normal distribution

$$b(k,n,p)\sim\frac{1}{\sqrt{2\pi np(1-p)}}\exp\frac{-(k-np)^2}{2np(1-p)}. \tag{A3}$$

To simplify notation, let us define $T=t+1$ and $M=m+1$. Now, using Eq. (A3) to approximate $b(T-j,MT,\rho)$ in Eq. (A1), and approximating sum by an integral, we obtain

$$P_t(0^{m+1})=\int_1^T\frac{x}{T}\frac{1}{\sqrt{2\pi MT\rho(1-\rho)}}\times\exp\left[\frac{(-T-x-MT\rho)^2}{2MT\rho(1-\rho)}\right]dx. \tag{A4}$$

Integration yields

$$P_t(0^{m+1})=\sqrt{\frac{M\rho(1-\rho)}{2\pi T}}\left\{\exp\left(\frac{-(1-T+M\rho T)^2}{2MT\rho(1-\rho)}\right)-\exp\left(\frac{-M\rho T}{2(1-\rho)}\right)\right\}+\frac{1}{2}(1-M\rho)\left\{\operatorname{erf}\left(\frac{M\rho T}{\sqrt{2M\rho(1-\rho)T}}\right)-\operatorname{erf}\left(\frac{1-T+M\rho T}{\sqrt{2M\rho(1-\rho)T}}\right)\right\},$$

where erf(x) denotes the error function

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt. \qquad \text{(A5)}$$

The first term in the above equation (involving two exponentials) tends to 0 with $T\to\infty$. Moreover, since $\lim_{x\to\infty}\mathrm{erf}(x)=1$, we obtain

$$\lim_{t\to\infty}P_t(0^{m+1})=\frac{1}{2}(1-M\rho)\times\left\{1-\lim_{T\to\infty}\mathrm{erf}\left(\frac{1-T+M\rho T}{\sqrt{2M\rho(1-\rho)T}}\right)\right\}.$$

Now, noting that

$$\lim_{T\to\infty}\mathrm{erf}\left(\frac{1-T+M\rho T}{\sqrt{2M\rho(1-\rho)T}}\right)=\begin{cases}1 & \text{if } M\rho\geqslant 1\\ -1 & \text{otherwise,}\end{cases} \qquad \text{(A6)}$$

and returning to the original notation, we recover Eq. (25):

$$\lim_{t\to\infty}P_t(0^{m+1})=\begin{cases}1-(m+1)\rho & \text{if } p<1/(m+1)\\ 0 & \text{otherwise.}\end{cases} \qquad \text{(A7)}$$

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